Lecture 6

Active Filters

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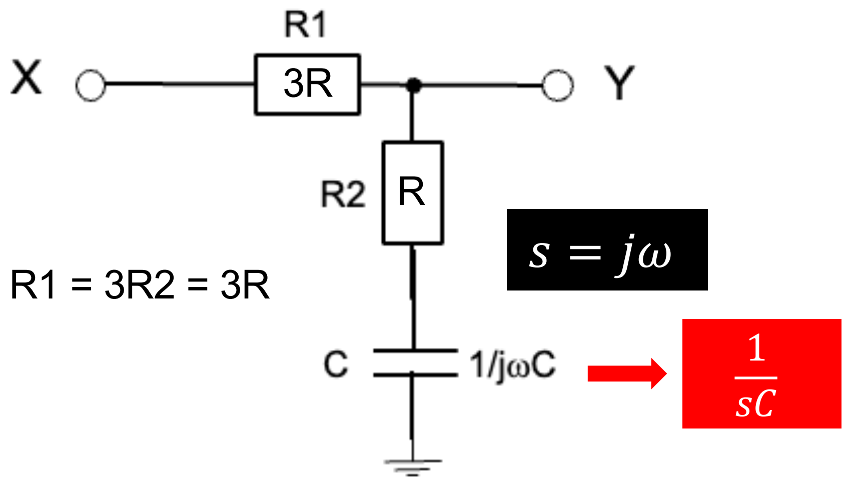
Transfer Function of 1st order LP Filter

From Year 1 ADC Part 1 Lecture 11, slide 3.



R1 = 3R2 = 3R

- More general if use **complex frequency s** to represent the quantity $j\omega$.
- Covered in Signals and Systems module this term, and Control Systems next term.
- Express impedance of capacitor as $\frac{1}{sC}$ instead of $\frac{1}{j\omega C}$.
- Capture both steady state (ac) and transient behaviour .

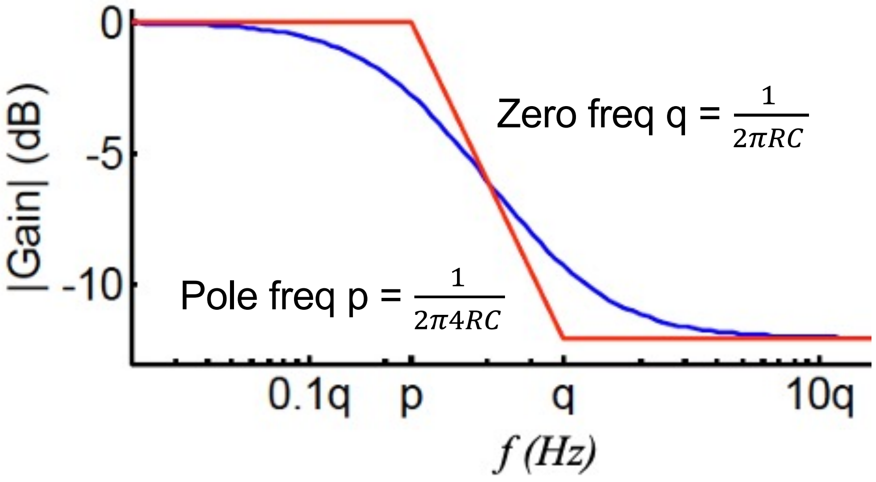
- Transfer function defined as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{R+1/sC}{4R+1/sC} = \frac{1+sRC}{1+4sRC}.$$

- Frequency response is calculated as

$$H(s)|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)} = \frac{1+j\omega RC}{1+4j\omega RC}.$$

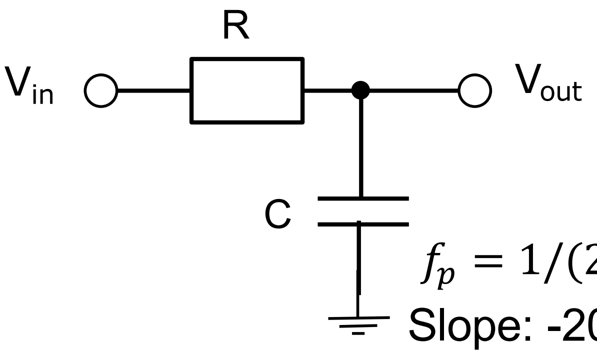
- Easier to perform algebra manipulation than using $j\omega$.
- Provides better intuitions on system behaviour.
- This simple filter is first-order low-pass with 1 pole and 1 zero.



- The break frequency occurs when real and imaginary parts are equal in numerator (zero freq) and denominator (pole freq).

1st order Active Filter

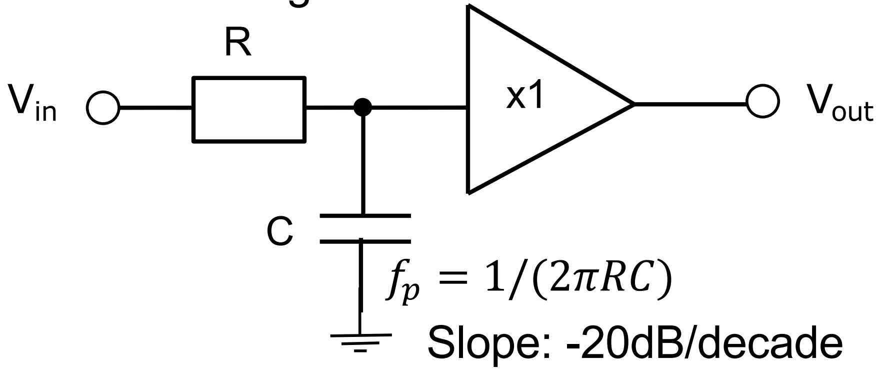
1st order passive filter
with 1 pole



$f_p = 1/(2\pi RC)$

Slope: -20dB/decade

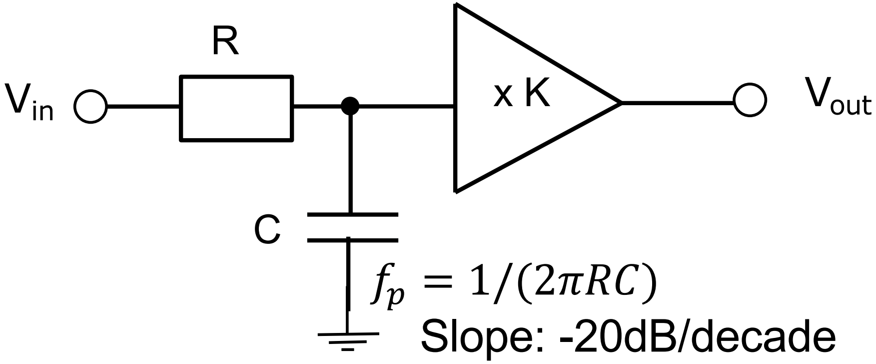
1st order active filter
gain of 1



$f_p = 1/(2\pi RC)$

Slope: -20dB/decade

1st order active filter
gain of K



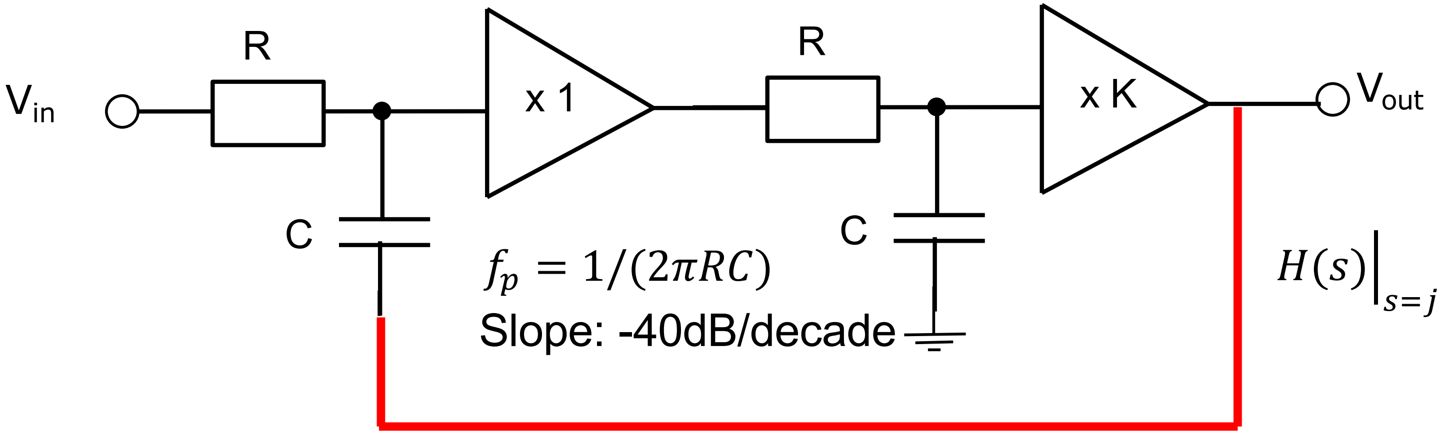
$f_p = 1/(2\pi RC)$

Slope: -20dB/decade

$$H(s)|_{s=j\omega} = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{K}{1+j\omega RC}.$$

2^{nd} order Active Lowpass Filter

2^{nd} order active filter with
2 poles and dc gain of K

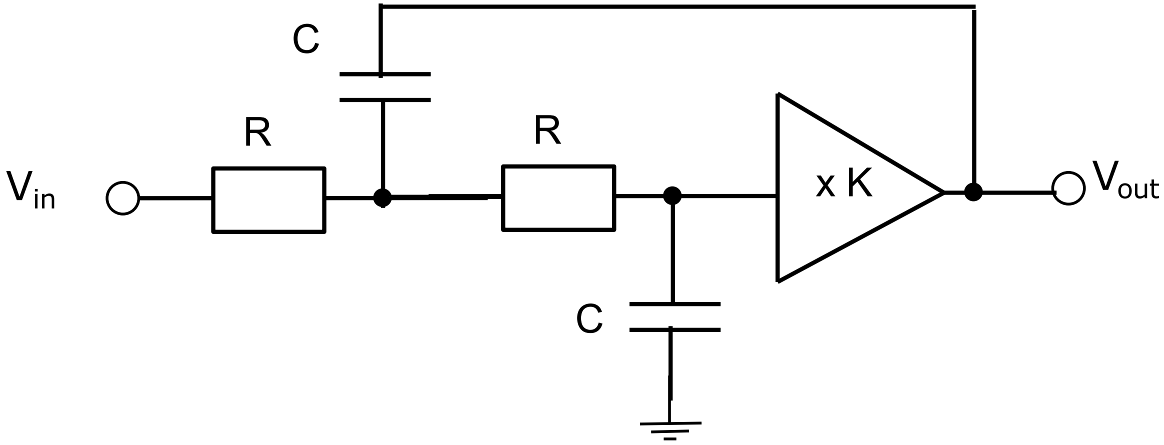


$f_p = 1/(2\pi RC)$

Slope: -40dB/decade

$$H(s)\Big|_{s=j\omega} = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{K}{(1+j\omega RC)^2}$$

2^{nd} order Sallen-Key Filter (1955)

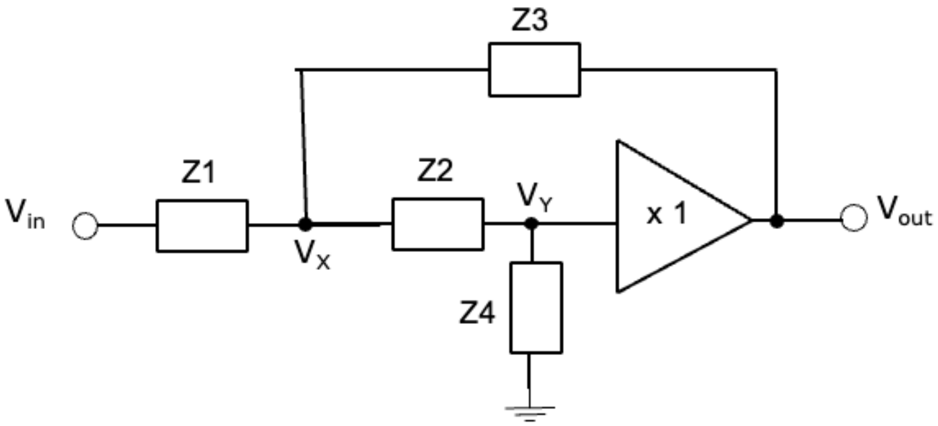


$f_p = 1/(2\pi RC)$

DC gain: K

Slope: -40dB/decade

Sallen-Key Filter Topology



- Invented by R.P. Sallen and E.L. Key in 1955 using valves as active devices (!)
- Z1 to Z4 are arbitrary impedance from resistors, capacitors or inductors.
- Assume amplifier gain is 1 (can be generalised to K), $V_Y = V_{out}$.
- Apply KCL to V_X yields:

$$\frac{V_{in} - V_x}{Z_1} + \frac{V_{out} - V_x}{Z_3} + \frac{V_{out} - V_x}{Z_4} = 0$$

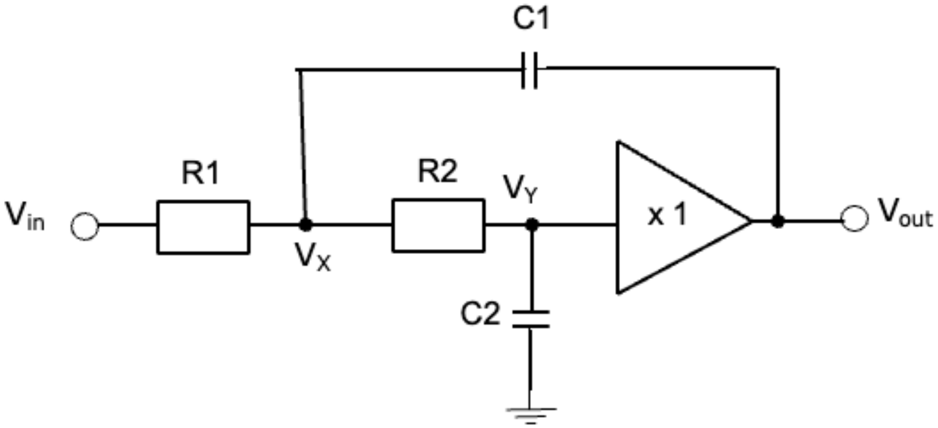
- Apply KCL at V_Y yields:

$$V_x = V_{out} + \frac{Z_2}{Z_4} V_{out} = V_{out}(1 + \frac{Z_2}{Z_4})$$

- Combining the two gives a general transfer function equation:

$$\frac{V_{out}}{V_{in}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1 + Z_2) + Z_3 Z_4}$$

2nd order Sallen-Key Lowpass Filter



$$\boxed{\frac{V_{out}}{V_{in}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1 + Z_2) + Z_3 Z_4}}$$

$$Z_1 = R_1, \quad Z_2 = R_2$$
$$Z_3 = 1/sC_1 \quad , \quad Z_4 = 1/sC_2$$

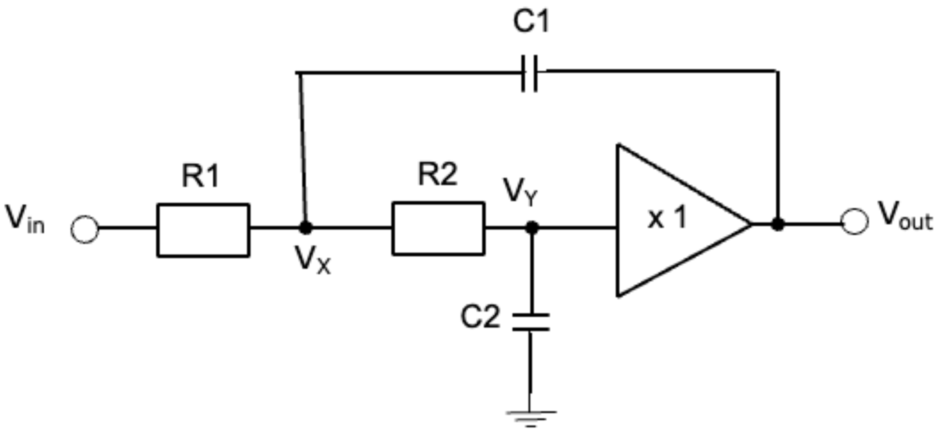
- Using the transfer function equation $H(s)$ from previous slice:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{s^2 C_1 C_2}}{R_1 R_2 + \frac{1}{sC_1}(R_1 + R_2) + \frac{1}{s^2 C_1 C_2}}$$

- Rearrange and put this in a standard form for a 2nd order lowpass system:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + \boldsymbol{C_2(R_1 + R_2)}\, s + \boldsymbol{C_1 C_2 R_1 R_2}\, s^2}$$

Significance of ω_0 and Q (1)



$$Z_1 = R_1, \quad Z_2 = R_2$$
$$Z_3 = 1/sC_1 \;, \quad Z_4 = 1/sC_2$$

- Put this into the standard form of a 2nd order lowpass filter is:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+\frac{1}{\omega_0 Q}s+\frac{1}{\omega_0^2}s^2} = \frac{1}{1+C_2(R_1+R_2)\,s+C_1C_2R_1R_2\,s^2}$$

Q is the quality factor
ζ is the damping ratio
$Q = \frac{1}{2\zeta}$

$$\frac{2\zeta}{\omega_0} = \frac{1}{\omega_0 Q}$$

$$\frac{1}{\omega_0^2}$$

ω_0 is the cutoff frequency (rad/s)

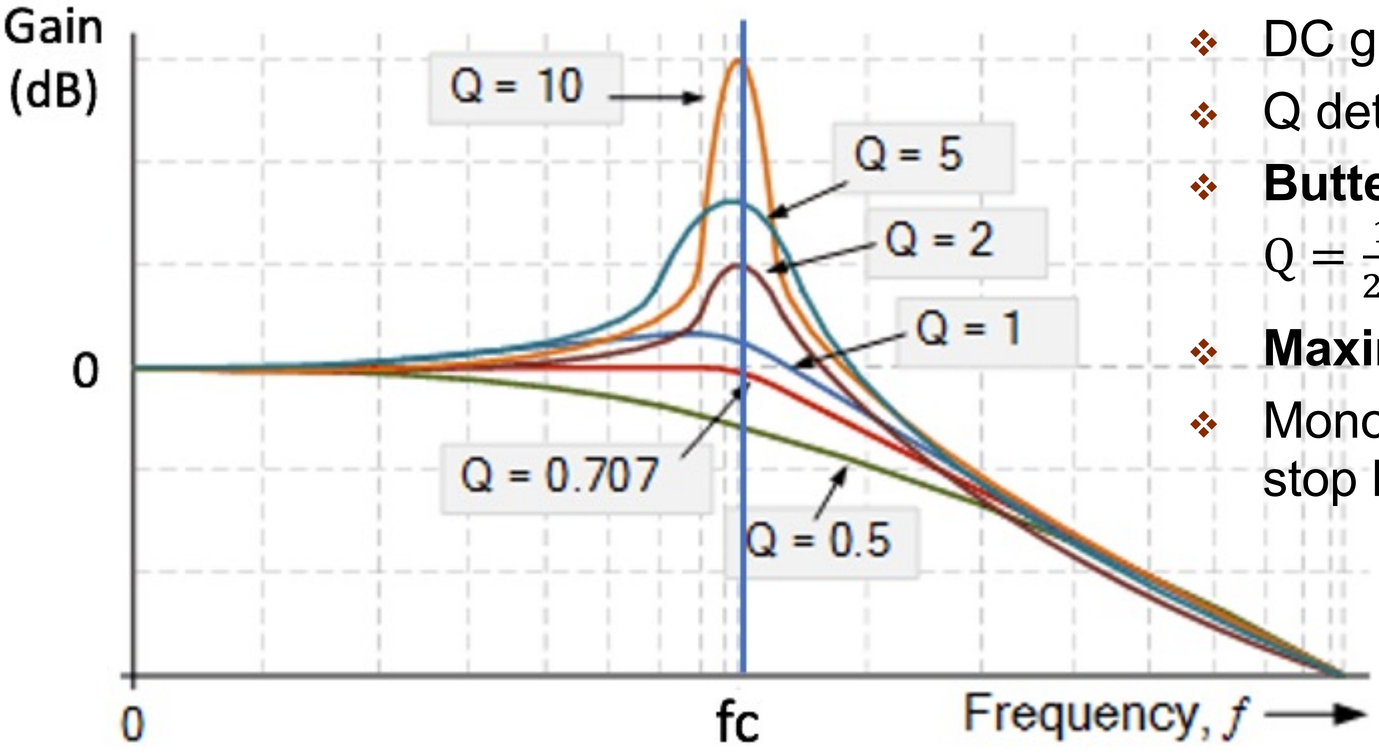
$f_c = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{C_1C_2R_1R_2}}$ in Hz

- Therefore, the cutoff frequency is: $f_c = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{C_1C_2R_1R_2}}$ Hz
- The quality factor Q is: $Q = \frac{\sqrt{C_1C_2R_1R_2}}{C_2(R_1+R_2)}$

Significance of ω_0 and Q (2)

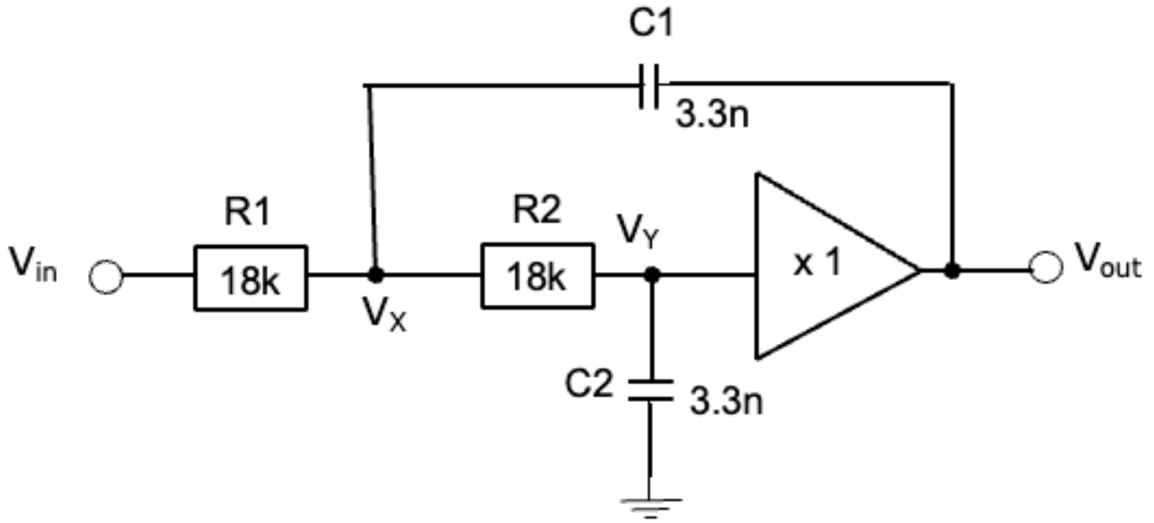
- Rewrite the transfer function as:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + \frac{1}{\omega_0 Q}s + \frac{1}{\omega_0^2}s^2} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad f_0 = \frac{\omega_0}{2\pi}$$



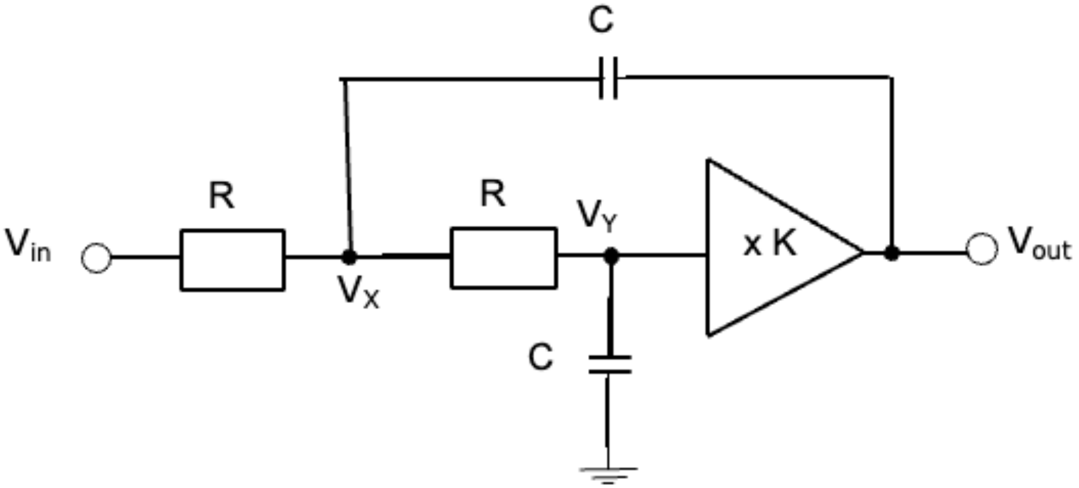
- ω_0 is the **cut-off frequency** of filter.
- DC gain of filter is 1 (i.e. s=0).
- Q determine how ‘peaky’ the filter is.
- **Butterworth filter**: $2\zeta = 1.414$, $\mathrm{Q} = \frac{1}{2\zeta} = 0.707$.
- **Maximally flat** gain in passband
- Monotonically decreasing gain in stop band.

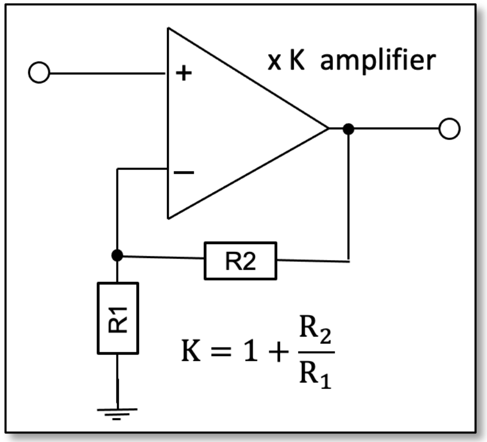
A simple Sallen-Key filter (from Lab 2)



- Simplify by making $R_1 = R_2 = R = 18k\Omega$, and $C_1 = C_2 = C = 3.3nF$.
- The cutoff frequency is: $f_c = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} = \frac{1}{2\pi RC} =$ 2.7kHz and
- $Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{C_2(R_1+R_2)} = \frac{RC}{(C\times 2R)} = \frac{1}{2}$.
- This is NOT a Butterworth filter because Q is not 0.707 or $\frac{1}{\sqrt{2}}$.

Sallen-Key filter with gain = K





- Keep same R and C values, fix Q by changing gain of op-amp K
- Left as an exercise to proof:

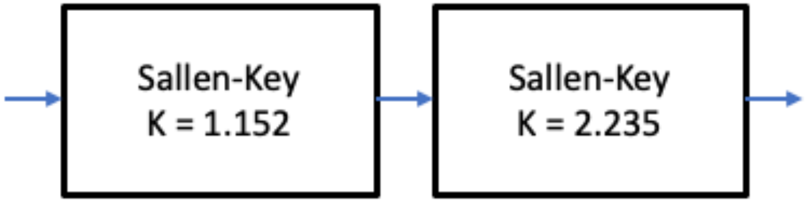
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = K \times \frac{1}{1 + \overbrace{(3-K)RC}^{\frac{1}{\omega_0 Q}}\, s + \overbrace{R^2C^2}^{\frac{1}{\omega_0^2}}\, s^2}$$

- Therefore, cutoff frequency f_c is same as before: $f_c = \frac{1}{2\pi\omega_0} = \frac{1}{2\pi RC}$.
- And, $Q = \frac{1}{\omega_0} \times \frac{1}{(3-K)RC} = \frac{1}{3-K}$.
- Therefore, to get a Butterworth filter with this topology, Q = 0.707, and
- $K = 3 - \frac{1}{Q} = 1.586.$

General Procedure: Butterworth LP filter

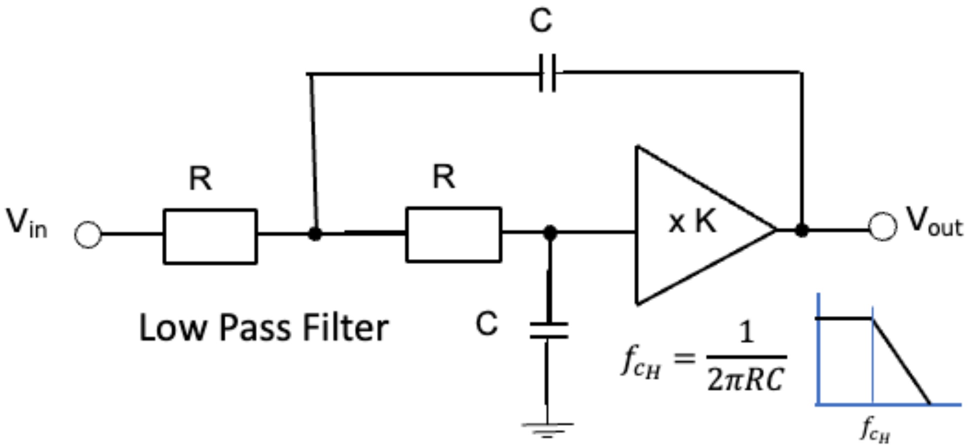
1. Determine the required cutoff frequency f_c.
2. Calculate R and C product with: $RC = \frac{1}{2\pi f_c}$.
3. Pick a suitable value of C >> input capacitance of op-amp (say in nF range).
4. Calculate value of R to give the required cutoff frequency.
5. Determine order of filter depending on required attenuation rate. Filter attenuation rate is -20 x n dB/decade, for an nth order filter.
6. Round n to the nearest high even number. You will need n/2 Salley-Key filter stages.
7. Use the table below to design gain of each stage of the filter. For example, for a 4th order Butterworth filter, we need two Sallen-Key stages with gain of 1.152 followed by 2.235.
8. Choose resistors for op-amps feedback paths to provides specified gain values.

4th order Butterworth LP Filter

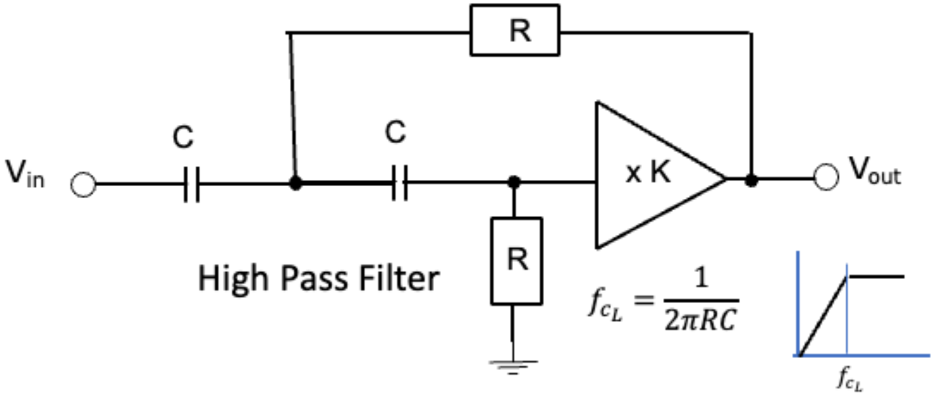


ORDER n	Gain values K
2	1.586
4	1.152, 2.235
6	1.068, 1.586, 2.483
8	1.038, 1.337, 1.889, 2.610

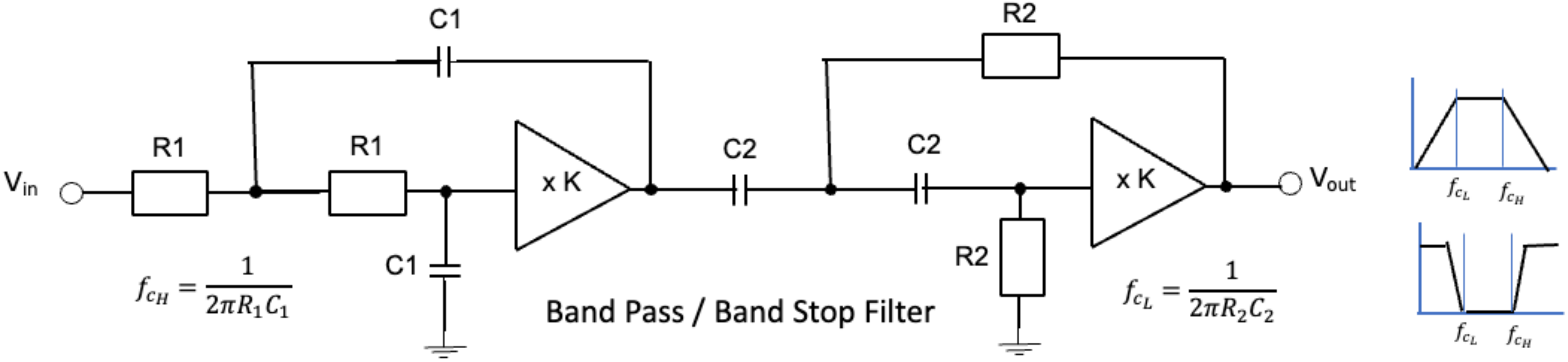
Other Sallen-Key filter circuits



$$f_{c_H} = \frac{1}{2\pi RC}$$



$$f_{c_L} = \frac{1}{2\pi RC}$$



$$f_{c_H} = \frac{1}{2\pi R_1 C_1}$$

$$f_{c_L} = \frac{1}{2\pi R_2 C_2}$$

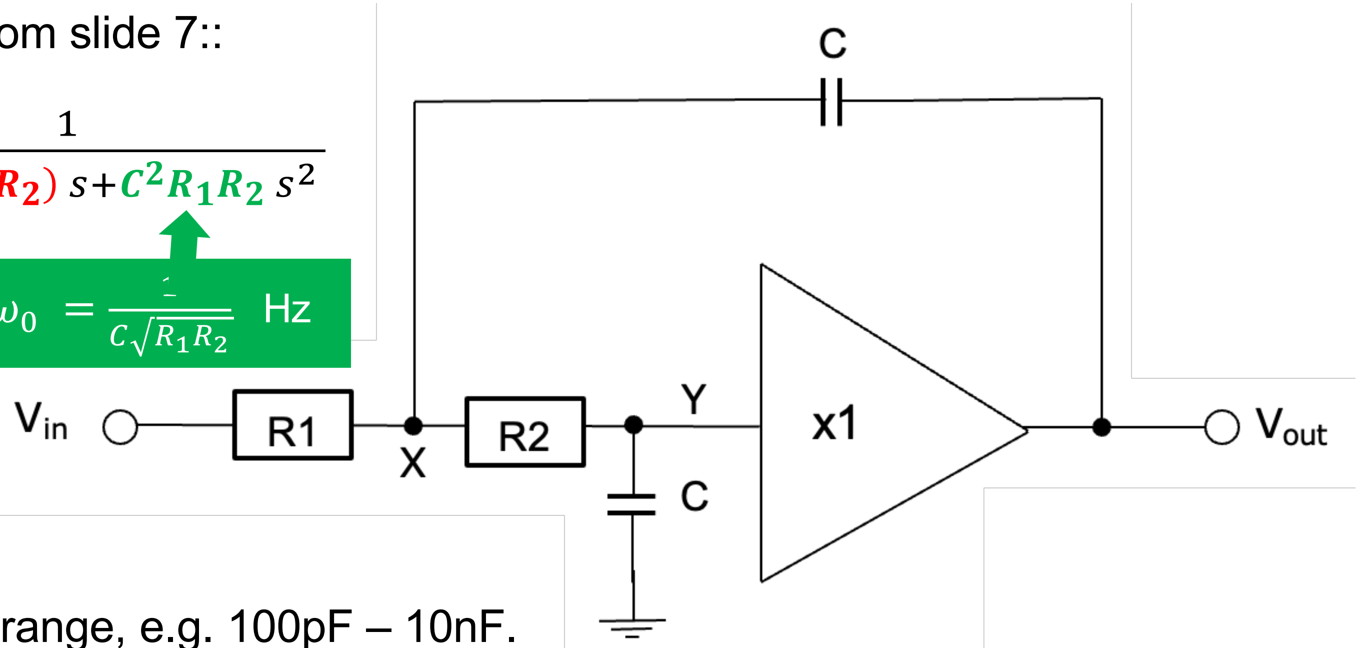
Using different values for R1 and R2

- Revisit transfer function of filter from slide 7::

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+\frac{1}{\omega_0 Q}s+\frac{1}{\omega_0^2}s^2} = \frac{1}{1+C(R_1+R_2)\,s+C^2R_1R_2\,s^2}$$

$$Q = \frac{\sqrt{R_1R_2}}{R_1+R_2}$$

$$\omega_0 = \frac{1}{C\sqrt{R_1R_2}} \text{ Hz}$$



- Design step:
  - Choose C in a reasonable range, e.g. 100pF – 10nF.
- Write down eq.1 for a given cutoff frequency.
- Write down eq. 2 for a given Q value.
- Solve the two equations for R1 and R2 values.